International Journal of Theoretical Physics, Vol. 46, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10773-006-9285-y

The Application of Extended tanh-Function Approach in Toda Lattice Equations

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Received August 6, 2006; accepted September 11, 2006
Published Online: February 27, 2007

In this paper, we generalize the extended tanh-function approach, which used to find new exact travelling wave solutions of nonlinear partial differential equations (NPDES) or coupled nonlinear partial differential equations, to nonlinear differential-difference equations (NDDES). As illustration, we discuss some Toda lattice equations, and solitary wave and periodic wave solutions of these Toda lattice equations are obtained by means of the extended tanh-function approach.

KEY WORDS: Toda lattice equations; extended tanh-function method.

PACS numbers: 05.45.Yv, 02.30.Jr, 02.30.Ik.

1. INTRODUCTION

Since the work of Fermi, Pasta, and Ulam in the 1950s (Fermi *et al.*, 1965), dynamical properties of the nonlinear differential-difference equations (NDDES) attract a growing interest due to their rich applicability in different physical problems such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, arrays of coupled Josephson's junctions, nonlinear fiber arrays, nonlinear charge and excitation transport in biological macromolecules, elastic energy transfer in anharmonic crystals, DNA molecule chains, etc. (Hickman and Hereman, 2003; Teschl, 2000). Unlike difference equations which are fully discretized, NDDES are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous.

Recently, there has been some interest in exactly soluble discretized nonlinear problem, and the inverse scattering method has been extended to a wide class of such problems (Ablowitz, 1978). More recently, with the development of symbolic computation, many direct and effective methods are presented to solve NDDES. For instance, Qian *et al.* (2003, 2004) have successfully extended the

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multilinear variable separation approach to special differential-difference equations. D. Baldwin *et al.* (2004) derived the kink-type solutions of many spatially discrete nonlinear models in terms of tanh function. Dai *et al.* (2004) obtained the kink-type solutions of the discrete sine-Gordon equation by means of the hyperbolic function approach. Zhu (2005) extended the tanh-function approach to solve $(2+1)$-dimensional Toda lattice equation and the discretized nonlinear mKdV lattice equation. Moreover, the Jacobian elliptic function method is generalized to solve differential-difference equations (Dai and Zhang, 2006).

However, these methods (Baldwin *et al.*, 2004; Dai *et al.*, 2004; Zhu, 2005; Dai and Zhang, 2006) with much complicated calculations can not give us an unified formulation to construct exact solutions. As an direct and simple methods, the extended tanh-function approach is extensively and successfully applied in many nonlinear partial differential equations to obtain exact solutions in a uniform way (Fan, 2000). Recently, Zheng *et al.* (2005) realized variable separation for dispersive long wave (DLW) equation via the extended tanh-function approach. Nevertheless, generalizing this method to solve differential-difference equations is quite hard because of the difficulty to search iterative relations between lattice indices, for example, the relations from indices n to $n \pm 1$. Fortunately, by careful analysis, we present the extended tanh-function method for differential-difference equations and successfully find the iterative relations between lattice indices. The virtue of this proposed method is that, without much complicated calculations, we circumvent integration to directly get many exact solutions in a uniform way. Another feature of this method is that it provides us a guideline to classify the various types of the solution according to the parameter δ. Applying this method, we investigate some Toda lattice equations and obtain some solitary wave and periodic wave solutions.

Our paper is organized as follows. In Section 2, the detailed extended tanh-function method for NDDES is given. In Section 3, we study some Toda lattice equations to illustrate the method. The last Section contains a short summary and discussion.

2. EXTENDED TANH-FUNCTION METHOD FOR NDDES

In this section, we would like to outline the extended tanh-function method for NDDES step by step.

Consider a system of M polynomial NDDES

$$\triangle(\mathbf{u_{n+p_1}}(\mathbf{x}), \ldots, \mathbf{u_{n+p_k}}(\mathbf{x}), \ldots, \mathbf{u}'_{\mathbf{n+p_1}}(\mathbf{x}), \ldots, \mathbf{u}'_{\mathbf{n+p_k}}(\mathbf{x}), \ldots, \mathbf{u}^{(\mathbf{r})}_{\mathbf{n+p_1}}(\mathbf{x}), \ldots, \mathbf{u}^{(\mathbf{r})}_{\mathbf{n+p_k}}(\mathbf{x})) = 0, \quad (1)$$

where the dependent variable $\mathbf{u_n}$ has M components $u_{i,n}$, the continuous variable $\mathbf{x}$ has N components x_i, the discrete variable $\mathbf{n}$ has Q components n_j, the k shift vectors $\mathbf{p}_i \in Z^Q$, and $\mathbf{u}^{(r)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

The main steps of the extended tanh-function method for NDDES are outlined as follows.

Step 1: When we seek the travelling wave solutions of Eq. (1), the first step is to introduce the wave transformation $\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x}) = \mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_\mathbf{n})$, $\xi_\mathbf{n} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta$ for any $s(s = 1, \ldots, k)$, where the coefficients $c_1, c_2, \ldots, c_N, d_1, d_2, \ldots, d_Q$ and the phase ζ are all constants. In this way, Eq. (1) becomes

$$\begin{aligned}&\triangle(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}), \ldots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n}), \ldots,\\ &\quad \mathbf{U}^{(\mathbf{r})}_{\mathbf{n}+\mathbf{p}_1}(\xi_\mathbf{n}), \ldots, \mathbf{U}^{(\mathbf{r})}_{\mathbf{n}+\mathbf{p}_k}(\xi_\mathbf{n})) = 0.\end{aligned} \tag{2}$$

Step 2: We propose the following series expansion as a solution of Eq. (2):

$$\mathbf{U}_\mathbf{n}(\xi_\mathbf{n}) = \sum_{j=-l}^{l} a_j \phi^j(\xi_\mathbf{n}), \tag{3}$$

where $\phi(\xi_\mathbf{n})$ satisfies the following Ricatti equation:

$$\frac{\mathrm{d}\phi(\xi_\mathbf{n})}{\mathrm{d}\xi_\mathbf{n}} = \delta + \phi^2(\xi_\mathbf{n}), \tag{4}$$

where δ is an arbitrary constant.

It is known that Eq. (4) possesses the solutions

$$\phi(\xi_\mathbf{n}) = \begin{cases} -\sqrt{-\delta}\tanh(\sqrt{-\delta}\xi_\mathbf{n}), \quad -\sqrt{-\delta}\coth(\sqrt{-\delta}\xi_\mathbf{n}), & \delta < 0, \\ \sqrt{\delta}\tan(\sqrt{\delta}\xi_\mathbf{n}), \quad -\sqrt{\delta}\cot(\sqrt{\delta}\xi_\mathbf{n}). & \delta > 0. \end{cases} \tag{5}$$

At present, one should note the identities

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}, \quad \coth(x+y) = \frac{\coth(x)+\tanh(y)}{1+\coth(x)\tanh(y)}, \tag{6}$$

and

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}, \quad \cot(x+y) = \frac{\cot(x)-\tan(y)}{1+\cot(x)\tan(y)}. \tag{7}$$

One can obviously rewrite the expressions (6) and (7) in an uniform formula by using of the expression (5)

$$\phi(\xi_\mathbf{n} + y) = \frac{\phi(\xi_\mathbf{n}) + \mu\sqrt{\mu\delta}\, f(\sqrt{\mu\delta}y)}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_\mathbf{n}) f(\sqrt{\mu\delta}y)}, \tag{8}$$

where $\mu = \pm 1$ and

$$f(\sqrt{\mu\delta}y) = \begin{cases} \tanh(\sqrt{-\delta}y), & \mu = -1, \\ \tan(\sqrt{\delta}y), & \mu = 1. \end{cases} \tag{9}$$

Thus

$$\mathbf{U_n}(\xi_{\mathbf{n}+\mathbf{p_s}}) = a_0 + \sum_{j=-l}^{l} a_j \left[\frac{\phi(\xi_{\mathbf{n}}) + \mu\sqrt{\mu\delta} f(\sqrt{\mu\delta}\varphi_{\mathbf{s}})}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_{\mathbf{n}}) f(\sqrt{\mu\delta}\varphi_{\mathbf{s}})} \right]^j, \tag{10}$$

where

$$\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q, \tag{11}$$

and p_{sj} is the j-th component of shift vector $\mathbf{p}_s$.

Step 3: Determine the degree l of the polynomial solutions (3) and (10). We are interested in balancing term $\phi(\xi_{\mathbf{n}})$, then the leading terms of $\mathbf{U_n}(\xi_{\mathbf{n}+\mathbf{p_s}})$, $(\mathbf{p_s} \neq 0)$ will not effect the balance since $\mathbf{U_n}(\xi_{\mathbf{n}+\mathbf{p_s}})$ can be interpreted as being of degree zero in $\phi(\xi_{\mathbf{n}})$. So we can easily get the degree l in the ansatzs (3) and (10) by balancing the highest nonlinear terms and the highest-order derivative term in $\mathbf{U_n}(\xi_{\mathbf{n}})$ as in the continuous case.

Step 4: Substituting the ansatzs (3) and (10) into Eq. (2), then setting the coefficients of all independent terms in $\phi(\xi_{\mathbf{n}})$ to zero, we will get a series of algebraic equations, from which the constants a_0, a_j $(j = 1, 2, \ldots, l)$ are explicitly determined by using of Maple and Wu elimination method.

Step 5: Substitute the values solved in Step 4 with Eq. (5) into expression (3), and one can find the solutions of Eq. (1). To assure the correctness of the solutions, it is necessary to substitute them into the original equation.

3. EXACT SOLUTIONS OF SOME TODA LATTICE EQUATIONS

In this section, we apply the method developed in the preceding section to some Toda lattice equations.

3.1. (2 + 1)-Dimensional Toda Lattice Equation

Consider the $(2+1)$-dimensional Toda lattice equation (Kajiwara and Satsuma, 1991)

$$\frac{\partial^2 y_n}{\partial x \partial t} = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}), \tag{12}$$

where $y_n(x, t)$ is the displacement from equilibrium of the n-th unit mass under an exponential decaying interaction force between nearest neighbors. With the change of variables $\frac{\partial u_n}{\partial t} = \exp(y_{n-1} - y_n) - 1$, lattice (12) can be written as

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}). \tag{13}$$

In this case, $\mathbf{u} = u$, $\mathbf{x} = \{x_1, x_2\} = \{t, x\}$, $\mathbf{n} = n_1 = n$ and $\mathbf{p_1} = p_1 = -1$, $\mathbf{p_2} = p_2 = 0$, $\mathbf{p_3} = p_3 = 1$, $d_1 = d$, $c_1 = c$, $c_2 = k$.

Now we assume that Eq. (13) has the solutions as ansatz (3). Then the balance procedure admit us to give the following formal travelling wave solutions of Eq. (13)

$$\begin{aligned} u_n &= a_0 + a_1\phi(\xi_n) + \frac{a_{-1}}{\phi(\xi_n)}, \\ u_{n+1} &= a_0 + a_1\frac{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + a_{-1}\frac{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}, \\ u_{n-1} &= a_0 + a_1\frac{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + a_{-1}\frac{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}, \end{aligned} \tag{14}$$

with

$$\xi_n = dn + kx + ct + \zeta, \tag{15}$$

where a_0, a_1, a_{-1}, d, k and c are constants to be determined later. Inserting the expression (14) along with (4) into Eq. (13); Clearing the denominator and eliminating the coefficients of independent terms in $\phi(\xi_n)$, yields

$$-2\,ca_1(a_1 + k)(f(\sqrt{\mu\,\delta}d))^2 = 0, \tag{16}$$

$$\begin{aligned} &2\,ca_1\mu(\delta\,a_1\mu^2 + \mu^2k\delta + b_1)(f(\sqrt{\mu\,\delta}d))^4 - 2\,a_1(a_1c\delta + a_1c\mu^2\delta \\ &\quad + ck\delta + 1 - ca_{-1})(f(\sqrt{\mu\,\delta}d))^2 + 2\,cka_1\mu\,\delta = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &2\,\mu\left(ca_1^2\delta^2\mu^4 + ca_1^2\mu^2\delta^2 + \delta\,a_1\mu^2 + \mu^2cka_1\delta^2 + a_{-1} - ca_{-1}^2\right. \\ &\quad \left.+ ca_{-1}a_1\delta\right)(f(\sqrt{\mu\,\delta}d))^4 - 2\,\delta\left(\delta\,\mu^2ca_1^2 + \delta\,cka_1\mu^4 + cka_{-1} + \mu^2a_1\right. \\ &\quad \left.- ca_1a_{-1}\right)(f(\sqrt{\mu\,\delta}d))^2 + 2\,cka_1\mu\,\delta^2 = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &2\,\mu\,\delta\left(ca_1^2\delta^2\mu^4 + \delta\,a_1\mu^4 - \delta\,\mu^4ca_1a_{-1} + \delta\,cka_{-1}\mu^2 - \mu^2ca_{-1}^2\right. \\ &\quad \left.+ a_{-1}\mu^2 - ca_{-1}^2\right)(f(\sqrt{\mu\,\delta}d))^4 - 2\,\delta\,(cka_1\mu^4\delta^2 + cka_{-1}\delta + \delta\,\mu^4ca_1a_{-1} \\ &\quad - \mu^2ca_{-1}^2 + a_{-1}\mu^2)(f(\sqrt{\mu\,\delta}d))^2 + 2\,ckb_1\mu\,\delta^2 = 0, \end{aligned} \tag{19}$$

$$2\,ca_{-1}\mu^3\delta^2(-\delta\, a_1\mu^2+\delta\, k-a_{-1})(f(\sqrt{\mu\,\delta}d))^4-2\,\delta^2\mu^2a_{-1}(ck\mu^2\delta+a_1c\mu^2\delta$$
$$-\mu^2ca_{-1}+\mu^2-ca_{-1}(f(\sqrt{\mu\,\delta}d))^2+2\,cka_{-1}\mu\,\delta^3=0, \tag{20}$$

$$-2\,c\mu^4\delta^3a_{-1}(\delta\, k-a_{-1})(f(\sqrt{\mu\,\delta}d))^2=0. \tag{21}$$

We solve the above system (16)–(21) by using the symbolic computation software *Maple* 10 and Wu's method, and obtain solutions of Eq. (13), namely,

$$u_n=a_0-\frac{\sqrt{-\delta}\sinh^2(\sqrt{-\delta}d)}{c}\tanh\left[\sqrt{-\delta}\left(dn-\frac{\sinh^2(\sqrt{-\delta}d)}{c}x+ct+\zeta\right)\right], \tag{22}$$

$$u_n=a_0-\frac{\sqrt{-\delta}\sinh^2(\sqrt{-\delta}d)}{c}\coth\left[\sqrt{-\delta}\left(dn-\frac{\sinh^2(\sqrt{-\delta}d)}{c}x+ct+\zeta\right)\right], \tag{23}$$

$$\begin{aligned}u_n=a_0-\frac{\sqrt{-\delta}\sinh^2(\sqrt{-\delta}d)\cosh^2(\sqrt{-\delta}d)}{c\delta}\Bigg\{\tanh\Bigg[\sqrt{-\delta}\Bigg(dn\\ -\frac{\sinh^2(\sqrt{-\delta}d)\cosh^2(\sqrt{-\delta}d)}{c}x+ct+\zeta\Bigg)\Bigg]\\ -\delta\coth\left[\sqrt{-\delta}\left(dn-\frac{\sinh^2(\sqrt{-\delta}d)\cosh^2(\sqrt{-\delta}d)}{c}x+ct+\zeta\right)\right]\Bigg\},\end{aligned} \tag{24}$$

$$u_n=a_0-\frac{\sqrt{\delta}\sin^2(\sqrt{\delta}d)}{\delta c}\tan\left[\sqrt{\delta}\left(dn+\frac{\sin^2(\sqrt{\delta}d)}{\delta c}x+ct+\zeta\right)\right], \tag{25}$$

$$u_n=a_0+\frac{\sqrt{\delta}\sin^2(\sqrt{\delta}d)}{\delta c}\cot\left[\sqrt{\delta}\left(dn+\frac{\sin^2(\sqrt{\delta}d)}{\delta c}x+ct+\zeta\right)\right], \tag{26}$$

$$\begin{aligned}u_n=a_0-\frac{\sqrt{\delta}\sin^2(\sqrt{\delta}d)\cos^2(\sqrt{\delta}d)}{c\delta}\\ \times\left\{\tan\left[\sqrt{\delta}\left(dn+\frac{\sin^2(\sqrt{\delta}d)\cos^2(\sqrt{\delta}d)}{c}x+ct+\zeta\right)\right]\right.\\ \left.-\delta\cot\left[\sqrt{\delta}\left(dn+\frac{\sin^2(\sqrt{\delta}d)\cos^2(\sqrt{\delta}d)}{c}x+ct+\zeta\right)\right]\right\},\end{aligned} \tag{27}$$

where a_0, d, c and ζ are arbitrary constants. Especially, when $\delta = -1$, the solution (22) degenerates as the solution in Ref. (Baldwin *et al.*, 2004).

3.2. Toda Lattice Equation

The Toda lattice equation (Toda, 1981) is of the form

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}u_n}{\mathrm{d}t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}). \tag{28}$$

With the same procedure in Section 3.1, we have the solutions of Eq. (28)

$$u_n = a_0 \pm \sinh(\sqrt{-\delta}d)\tanh\left[\sqrt{-\delta}\left(dn \pm \sqrt{-\frac{1}{\delta}}\sinh(\sqrt{-\delta}d)t + \zeta\right)\right], \tag{29}$$

$$u_n = a_0 \pm \sinh(\sqrt{-\delta}d)\coth\left[\sqrt{-\delta}\left(dn \pm \sqrt{-\frac{1}{\delta}}\sinh(\sqrt{-\delta}d)t + \zeta\right)\right], \tag{30}$$

$$\begin{aligned} u_n &= a_0 \pm \sinh(\sqrt{-\delta}d)\cosh(\sqrt{-\delta}d) \\ &\quad \times \left\{ \tanh\left[\sqrt{-\delta}(dn \pm \sqrt{-\frac{1}{\delta}}\sinh(\sqrt{-\delta}d)\cosh(\sqrt{-\delta}d)t + \zeta)\right]\right. \\ &\quad \left. - \delta\coth\left[\sqrt{-\delta}\left(dn \pm \sqrt{-\frac{1}{\delta}}\sinh(\sqrt{-\delta}d)\cosh(\sqrt{-\delta}d)t + \zeta\right)\right]\right\}, \end{aligned} \tag{31}$$

$$u_n = a_0 \pm \sin(\sqrt{\delta}d)\tan\left[\sqrt{\delta}\left(dn \mp \sqrt{\frac{1}{\delta}}\sin(\sqrt{\delta}d)t + \zeta\right)\right], \tag{32}$$

$$u_n = a_0 \pm \sin(\sqrt{\delta}d)\cot\left[\sqrt{\delta}\left(dn \pm \sqrt{\frac{1}{\delta}}\sin(\sqrt{\delta}d)t + \zeta\right)\right], \tag{33}$$

$$\begin{aligned} u_n &= a_0 \pm \sin(\sqrt{\delta}d)\cos(\sqrt{\delta}d)\left\{ \tan\left[\sqrt{\delta}(dn \mp \sqrt{\frac{1}{\delta}}\sin(\sqrt{\delta}d)\cos(\sqrt{\delta}d)t + \zeta)\right]\right. \\ &\quad \left. - \delta\cot\left[\sqrt{\delta}\left(dn \mp \sqrt{\frac{1}{\delta}}\sin(\sqrt{\delta}d)\cos(\sqrt{\delta}d)t + \zeta\right)\right]\right\}, \end{aligned} \tag{34}$$

where a_0, d and ζ are arbitrary constants. Especially, when $\delta = -1$, the solution (29) degenerates as the solution in Ref. (Baldwin *et al.*, 2004).

3.3. Another Toda Lattice Equation

Another Toda lattice equation (Suris, 1997) has the form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_n(v_n - v_{n-1}), \quad \frac{\mathrm{d}v_n}{\mathrm{d}t} = v_n(u_{n+1} - u_n). \tag{35}$$

In this case, $\mathbf{u} = \{u, v\}$, $\mathbf{x} = x_1 = t$, $\mathbf{n} = n_1 = n$ and $\mathbf{p_1} = p_1 = -1$, $\mathbf{p_2} = p_2 = 0$, $\mathbf{p_3} = p_3 = 1$, $d_1 = d$, $c_1 = c$.

The balance procedure admit us to give the following formal travelling wave solutions of Eq. (35)

$$\begin{aligned}
u_n &= a_0 + a_1\phi(\xi_n) + \frac{a_{-1}}{\phi(\xi_n)}, \quad v_n = b_0 + b_1\phi(\xi_n) + \frac{b_{-1}}{\phi(\xi_n)},\\
u_{n+1} &= a_0 + a_1\frac{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + a_{-1}\frac{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)},\\
v_{n+1} &= b_0 + b_1\frac{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + b_{-1}\frac{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) + \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)},\\
u_{n-1} &= a_0 + a_1\frac{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + a_{-1}\frac{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)},\\
v_{n-1} &= b_0 + b_1\frac{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)}{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)} + b_{-1}\frac{1 + \frac{1}{\sqrt{\mu\delta}}\phi(\xi_n)f(\sqrt{\mu\delta}d)}{\phi(\xi_n) - \mu\sqrt{\mu\delta}f(\sqrt{\mu\delta}d)},
\end{aligned} \tag{36}$$

with

$$\xi_n = dn + ct + \zeta, \tag{37}$$

where $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, d$ and c are constants to be determined later. Inserting the expression (36) along with (4) into Eq. (35); Clearing the denominator and eliminating the coefficients of independent terms in $\phi(\xi_n)$, yields a series of equations, from which we have the solutions of Eq. (35)

$$\begin{aligned}
u_n &= -c\sqrt{-\delta}\coth(\sqrt{-\delta}d) + c\sqrt{-\delta}\tanh[\sqrt{-\delta}(dn + ct + \zeta)],\\
v_n &= -c\sqrt{-\delta}\coth(\sqrt{-\delta}d) - c\sqrt{-\delta}\tanh[\sqrt{-\delta}(dn + ct + \zeta)],
\end{aligned} \tag{38}$$

$$\begin{aligned}
u_n &= -c\sqrt{-\delta}\coth(\sqrt{-\delta}d) + c\sqrt{-\delta}\coth[\sqrt{-\delta}(dn + ct + \zeta)],\\
v_n &= -c\sqrt{-\delta}\coth(\sqrt{-\delta}d) - c\sqrt{-\delta}\coth[\sqrt{-\delta}(dn + ct + \zeta)],
\end{aligned} \tag{39}$$

$$
\begin{aligned}
u_n &= -c\sqrt{-\delta}[\tanh(\sqrt{-\delta}d) + \coth(\sqrt{-\delta}d)] + c\sqrt{-\delta}\{\tanh[\sqrt{-\delta}(dn + ct + \zeta)] \\
&\quad - \delta\coth[\sqrt{-\delta}(dn + ct + \zeta)]\}, \\
v_n &= -c\sqrt{-\delta}[\tanh(\sqrt{-\delta}d) + \coth(\sqrt{-\delta}d)] - c\sqrt{-\delta}\{\tanh[\sqrt{-\delta}(dn + ct + \zeta)] \\
&\quad + \delta\coth[\sqrt{-\delta}(dn + ct + \zeta)]\}, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
u_n &= -c\sqrt{-\delta}\cot(\sqrt{\delta}d) - c\sqrt{\delta}\tan[\sqrt{\delta}(dn + ct + \zeta)], \\
v_n &= -c\sqrt{\delta}\cot(\sqrt{\delta}d) + c\sqrt{\delta}\tan[\sqrt{-\delta}(dn + ct + \zeta)], \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
u_n &= -c\sqrt{\delta}\cot(\sqrt{\delta}d) + c\sqrt{\delta}\cot[\sqrt{\delta}(dn + ct + \zeta)], \\
v_n &= -c\sqrt{\delta}\cot(\sqrt{\delta}d) - c\sqrt{\delta}\cot[\sqrt{\delta}(dn + ct + \zeta)], \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
u_n &= c\sqrt{\delta}[\tan(\sqrt{\delta}d) - \cot(\sqrt{\delta}d)] - c\sqrt{\delta}\{\tan[\sqrt{\delta}(dn + ct + \zeta)] \\
&\quad - \delta\cot[\sqrt{\delta}(dn + ct + \zeta)]\}, \\
v_n &= c\sqrt{\delta}[\tan(\sqrt{\delta}d) - \cot(\sqrt{\delta}d)] + c\sqrt{\delta}\{\tan[\sqrt{\delta}(dn + ct + \zeta)] \\
&\quad - \delta\cot[\sqrt{\delta}(dn + ct + \zeta)]\}, \qquad (43)
\end{aligned}
$$

where d, c and ζ are arbitrary constants. Especially, when $\delta = -1$, the solution (38) degenerates as the solution in Ref. (Baldwin *et al.*, 2004).

3.4. Relativistic Toda Equation

Relativistic Toda equation (Suris, 1998) reads

$$
\frac{\mathrm{d}u_n}{\mathrm{d}t} = (1 + \alpha u_n)(v_n - v_{n-1}),
$$

$$
\frac{\mathrm{d}v_n}{\mathrm{d}t} = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \qquad (44)
$$

With the same procedure in Section 3.3, we have the solutions of Eq. (44), i.e.

$$
\begin{aligned}
u_n &= -\frac{1}{\alpha} - c\sqrt{-\delta}\coth(\sqrt{-\delta}d) + c\sqrt{-\delta}\tanh[\sqrt{-\delta}(dn + ct + \zeta)], \\
v_n &= \frac{c\sqrt{-\delta}}{\alpha}\coth(\sqrt{-\delta}d) - \frac{c\sqrt{-\delta}}{\alpha}\tanh[\sqrt{-\delta}(dn + ct + \zeta)], \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
u_n &= -\frac{1}{\alpha} - c\sqrt{-\delta}\coth(\sqrt{-\delta}d) + c\sqrt{-\delta}\coth[\sqrt{-\delta}(dn + ct + \zeta)], \\
v_n &= \frac{c\sqrt{-\delta}}{\alpha}\coth(\sqrt{-\delta}d) - \frac{c\sqrt{-\delta}}{\alpha}\coth[\sqrt{-\delta}(dn + ct + \zeta)], \qquad (46)
\end{aligned}
$$

$$u_n = -\frac{1}{\alpha} - c\sqrt{-\delta}[\tanh(\sqrt{-\delta}d) + \coth(\sqrt{-\delta}d)]$$
$$+ c\sqrt{-\delta}\{\tanh[\sqrt{-\delta}(dn + ct + \zeta)] - \delta\coth[\sqrt{-\delta}(dn + ct + \zeta)]\},$$
$$v_n = \frac{c\sqrt{-\delta}}{\alpha}[\tanh(\sqrt{-\delta}d) + \coth(\sqrt{-\delta}d)] - \frac{c\sqrt{-\delta}}{\alpha}\{\tanh[\sqrt{-\delta}(dn + ct + \zeta)]$$
$$- \delta\coth[\sqrt{-\delta}(dn + ct + \zeta)]\}, \quad (47)$$
$$u_n = -\frac{1}{\alpha} - c\sqrt{\delta}\cot(\sqrt{\delta}d) - c\sqrt{\delta}\tan[\sqrt{\delta}(dn + ct + \zeta)],$$
$$v_n = \frac{c\sqrt{\delta}}{\alpha}\cot(\sqrt{\delta}d) + \frac{c\sqrt{\delta}}{\alpha}\tan[\sqrt{\delta}(dn + ct + \zeta)], \quad (48)$$
$$u_n = -\frac{1}{\alpha} - c\sqrt{\delta}\cot(\sqrt{\delta}d) + c\sqrt{\delta}\cot[\sqrt{\delta}(dn + ct + \zeta)],$$
$$v_n = \frac{c\sqrt{\delta}}{\alpha}\cot(\sqrt{\delta}d) - \frac{c\sqrt{\delta}}{\alpha}\cot[\sqrt{\delta}(dn + ct + \zeta)], \quad (49)$$
$$u_n = -\frac{1}{\alpha} + c\sqrt{\delta}[\tan(\sqrt{\delta}d) - \cot(\sqrt{\delta}d)] - c\sqrt{\delta}\{\tan[\sqrt{\delta}(dn + ct + \zeta)]$$
$$- \delta\cot[\sqrt{\delta}(dn + ct + \zeta)]\},$$
$$v_n = -\frac{c\sqrt{\delta}}{\alpha}[\tan(\sqrt{\delta}d) - \cot(\sqrt{\delta}d)] + \frac{c\sqrt{\delta}}{\alpha}\{\tan[\sqrt{\delta}(dn + ct + \zeta)]$$
$$- \delta\cot[\sqrt{\delta}(dn + ct + \zeta)]\}, \quad (50)$$

where d, c and ζ are arbitrary constants. Especially, when $\delta = -1$, the solution (45) degenerates as the solution in Ref. (Baldwin *et al.*, 2004).

4. SUMMARY AND DISCUSSION

In conclusion, we have utilized the extended tanh-function approach to construct solitary wave and periodic wave solutions of some Toda lattice equations in a uniform way without much complicated calculations. The polynomial solutions of NDDES in tanh, coth, tan and cot have been derived. Although these solutions are only a small part of the large variety of possible solutions for these equations discussed here, they might serve as seeding solutions for a class of localized structures which exist in these systems. We hope that they will be useful in further perturbative and numerical analysis of various solutions to these Toda equations.

This method presented in this paper is only an initial work, more work will be done. The more applications of this method to other nonlinear differential-difference systems deserve further investigation.

In principle, we naturally present a more general ansatzs, which read

$$\mathbf{U_n}(\xi_\mathbf{n}) = \sum_{j=-l}^{l} [a_j \phi^j(\xi_\mathbf{n}) + b_j \phi^{j-1}(\xi_\mathbf{n})\sqrt{\delta + \phi^2(\xi_\mathbf{n})}], \tag{51}$$

$$\begin{aligned}\mathbf{U_n}(\xi_{\mathbf{n}+\mathbf{p_s}}) = \sum_{j=-l}^{l} \Bigg\{ & a_j \left[\frac{\phi(\xi_\mathbf{n}) + \mu\sqrt{\mu\delta} f(\sqrt{\mu\delta}\varphi_\mathbf{s})}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_\mathbf{n}) f(\sqrt{\mu\delta}\varphi_\mathbf{s})}\right]^j \\ & + b_j \left[\frac{\phi(\xi_\mathbf{n}) + \mu\sqrt{\mu\delta} f(\sqrt{\mu\delta}\varphi_\mathbf{s})}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_\mathbf{n}) f(\sqrt{\mu\delta}\varphi_\mathbf{s})}\right]^{j-1} \\ & \times \sqrt{\delta^2 + \left[\frac{\phi(\xi_\mathbf{n}) + \mu\sqrt{\mu\delta} f(\sqrt{\mu\delta}\varphi_\mathbf{s})}{1 - \frac{1}{\sqrt{\mu\delta}}\phi(\xi_\mathbf{n}) f(\sqrt{\mu\delta}\varphi_\mathbf{s})}\right]^2}\Bigg\},\end{aligned} \tag{52}$$

where $\phi(\xi_\mathbf{n})$ satisfies the Ricatti Eq. (4) and φ_s is given in Eq. (11), a_j, b_j $(j = -l, \ldots, 1, 2, \ldots, l)$ are constants which need to determining. The application of this general ansatzs to nonlinear differential-difference systems will be also studied further.

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